PHYSICAL REVIEW E **66**, 050102(R) (2002)

Kinetics of migration-driven aggregation processes

Jianhong Ke* and Zhenquan Lin
Department of Physics, Wenzhou Normal College, Wenzhou 325027, China

(Received 27 June 2002; revised manuscript received 14 August 2002; published 19 November 2002)

We study the kinetic behavior of the growth of aggregates driven by reversible migration between any two aggregates. For the simple system with the migration rate kernel $K(k;j)=K'(k;j)\propto kj^{\upsilon}$ at which the monomers migrate between the aggregates of size k and those of size j, we find that for the $\upsilon\leqslant 2$ case the evolution of the system always obeys a scaling law. Moreover, the typical aggregate size grows as $\exp(2IA_0t)$ in the case of $\upsilon=2$ and as $t^{1/(2-\upsilon)}$ in the case of $-1<\upsilon<2$. In particular, when $\upsilon\leqslant -2$, the typical aggregate size always grows as $t^{1/3}$ and the aggregate-size distribution approaches a similar scaling form.

DOI: 10.1103/PhysRevE.66.050102 PACS number(s): 82.20.−w, 05.40.−a, 68.43.Jk, 89.75.Da

Irreversible aggregation is an important phenomenon in natural science [1–3], and considerable amount of works have helped to understand the kinetics of these aggregation processes well [4–8]. Recently, much interest has been devoted to linking the discussions about the natural aggregation phenomena to those in life science, such as sociology and economy [9–12]. In order to study universal aggregation mechanisms, Ispolatov *et al.* introduced several different asset exchange models for the evolution of the wealth distribution in the economic interaction population [13], and Leyvraz and Redner proposed a migration-driven aggregate growth model for the evolution of city populations [14]. In these models, there exists preferential evaporation from smaller aggregates and preferential condensation onto larger aggregates, which is described by an irreversible reaction scheme, $A_k+A_l \xrightarrow{K(k;l)} A_{k-1}+A_{l+1}$ $(k\leqslant l)$. Here, A_k denotes an aggregate consisting of size k and $K(k;l)$ is the migration rate dependent on the sizes of the reactants. The solution to the rate equation exhibited that the kinetics of this process obeys a very different scaling law from that of the conventional aggregation process. In fact, the class of the migration-driven aggregation phenomena occurs in many branches of physics and social sciences [15]. However, for some actual cases the migration direction may not depend on the relative sizes of the two aggregates. In this paper, we investigate the kinetics of a general migration-driven aggregation model, in which migration goes from the larger to the smaller aggregates as well as from the smaller to the larger ones.

We now define our migration-driven aggregation model. The aggregate A_k of size k loses a monomer with a rate $K(k;j)$ according to the emigration scheme $A_k+A_j \xrightarrow{K(k;j)} A_{k-1}+A_j^{+1}$, where A_j^{+1} represents an aggregate A_j containing a positively charged monomer. Meanwhile, the aggregate A_l gains a monomer with a rate $K'(l;j')$ according to the immigration scheme $A_l+A_{j'} \xrightarrow{K'(l;j')} A_{l+1}+A_{j'}^{-1}$, where $A_{j'}^{-1}$ denotes the $A_{j'}$ aggregate with a negatively charged monomer. On the other hand, we assume that a positively charged aggregate and a negatively charged aggregate react immediately in the scheme $A_j^{+1}+A_{j'}^{-1} \xrightarrow{J(j;j')} A_j+A_{j'}$, with the rate $J(j;j')\rightarrow\infty$. The migration scheme of our model is thus equivalent to the reaction $A_k+A_l\rightarrow A_{k-1}+A_{l+1}$, and the aggregates A_j and $A_{j'}$ in the above reactions play roles analogous to the monomer sink and source, respectively. It is believed that our model may mimic some social and economic processes. For example, one person leaves his hometown (population center) and emigrates to another district that may not admit him; thus we can consider that the later district contains a "positively charged" population. On the other hand, a district that is deprived of a staff member by another one leaves an unoccupied position. If possible, an "unnecessary" person may easily take over the vacant position. This phenomenon may occur frequently in a commercially developed area.

We assume that the system has spatial homogeneity, so that the fluctuations in the densities of the reactants are ignored and the aggregates are considered to be homogeneously distributed in the space throughout the process. Thus, the theoretical approach to investigate the kinetics of the aggregation process can be based on the mean-field rate equation, which assumes that the reaction proceeds with a rate proportional to the reactant concentrations. Let $a_k(t)$ be the concentration of aggregates A_k of size k at time t. By generalizing the rate equation of the migration-driven aggregation process given by Ref. [14], we write the corresponding rate equation for our system as follows:

$$\frac{da_k}{dt}=\sum_{j=1}^{\infty} K(k+1;j)a_{k+1}a_j+\sum_{j=1}^{\infty} K'(k-1;j)a_{k-1}a_j -\sum_{j=1}^{\infty}[K(k;j)+K'(k;j)]a_ka_j, \tag{1}$$

where we impose the boundary condition $a_0(t)=0$.

For simplicity, we consider a model with a symmetrical migration rate kernel. The rate of the A_k aggregate gaining or losing one monomer is directly proportional to its size k and to j^{υ} of the opposite aggregate A_j, i.e., $K(k;j)=K'(k;j)=Ikj^{\upsilon}$ (I is a constant). Then Eq. (1) reduces to

$$\frac{da_k}{dt}=IM_{\upsilon}(t)[(k+1)a_{k+1}+(k-1)a_{k-1}-2ka_k], \tag{2}$$

*Electronic address: kejianhong@yahoo.com.cn

where $M_\upsilon(t)=\Sigma_{j=1}^{\infty}j^{\upsilon}a_j(t)$.

We assume that there only exist the monomer aggregates at $t=0$ and the concentration is equal to A_0. Then the initial condition is $a_k(0)=A_0\delta_{k1}$. We first consider several simple cases with integral index υ.

When $\upsilon=0$, Eq. (2) can be solved with the help of ansatz [16]

$$a_k(t)=A(t)[a(t)]^{k-1}. \tag{3}$$

Substituting Eq. (3) into Eq. (2), we can transform the rate equation (2) into the following differential equations:

$$\frac{da}{dt}=IA(1-a) \quad \frac{dA}{dt}=-2IA^2, \tag{4}$$

with the corresponding initial conditions

$$a=0, \quad A=A_0 \quad \text{at} \ \ t=0. \tag{5}$$

One can then derive the exact solutions of $a(t)$ and $A(t)$ from Eqs. (4):

$$a(t)=1-(2IA_0t+1)^{-1/2}, \quad A(t)=A_0(2IA_0t+1)^{-1}. \tag{6}$$

Thus we obtain the exact solution of $a_k(t)$:

$$\begin{aligned} a_k(t)&=A(t)[a(t)]^{k-1}\\ &=A_0(2IA_0t+1)^{-1}[1-(2IA_0t+1)^{-1/2}]^{k-1}. \end{aligned} \tag{7}$$

Further, Eq. (7) can be rewritten as follows:

$$a_k(t)\simeq(2It)^{-1}\exp(-x), \tag{8}$$

which is valid in the scaling region

$$k\gg1, \quad t\gg1, \quad x=k(2IA_0t)^{-1/2}=\text{finite}. \tag{9}$$

This implies that in the long-time limit the asymptotic aggregate-size distribution satisfies the scaling form [16]

$$a_k(t)\simeq t^{-w}\Phi[k/S(t)], \quad S(t)\propto t^z, \tag{10}$$

where $S(t)$ is the characteristic aggregate size of such an aggregation process. Here, the scaling function is exponential, $\Phi(x)=\exp(-x)$, and the governing exponents are $w=1$ and $z=1/2$. The total number is $M_0(t)=\Sigma_{k=1}^{\infty}a_k(t)=A_0(2IA_0t+1)^{-1/2}$, which implies that the total density decays as $t^{-1/2}$. On the other hand, the total mass of the aggregates, $M_1(t)=\Sigma_{k=1}^{\infty}ka_k(t)\equiv A_0$, is conserved naturally by the dynamics of the migration-driven aggregation process under any initial conditions (not necessarily the monodisperse one). We also find that for this case the typical aggregate size $S(t)$ grows as $t^{1/2}$.

When $\upsilon=1$, with the help of ansatz (3) we then obtain the following equations from Eq. (2):

$$\frac{da}{dt}=IA, \quad \frac{dA}{dt}=-\frac{2IA^2}{1-a}. \tag{11}$$

Equations (11) are directly solved to yield

$$a(t)=1-(IA_0t+1)^{-1}, \quad A(t)=A_0(IA_0t+1)^{-2}. \tag{12}$$

Thus we obtain the scaling solution of $a_k(t)$ in the long-time limit:

$$a_k(t)\simeq I^{-2}A_0^{-1}t^{-2}\exp(-x), \quad x=k(IA_0t)^{-1}, \tag{13}$$

with the exponents $w=2$ and $z=1$. In this case, the total number decays as t^{-1} and the typical aggregate size grows as t. Moreover, the total mass is conserved by the dynamics of the system. The results also imply that this case is equivalent to the general irreversible aggregation with a constant reaction rate.

When $\upsilon=2$, by employing the above technique we recast Eq. (2) to

$$\frac{da}{dt}=IA+\frac{2IaA}{1-a}, \quad \frac{dA}{dt}=-\frac{2IA^2}{1-a}-\frac{4IaA^2}{(1-a)^2}. \tag{14}$$

These directly yield

$$\begin{aligned} a(t)&=1-2[\exp(2IA_0t)+1]^{-1},\\ A(t)&=4A_0[\exp(2IA_0t)+1]^{-2}. \end{aligned} \tag{15}$$

In the long-time limit, the scaling solution of $a_k(t)$ is then given as follows:

$$a_k(t)\simeq 4A_0e^{-4IA_0t}\exp(-x), \quad x=2ke^{-2IA_0t}. \tag{16}$$

This satisfies the generalized scaling form [17]

$$a_k(t)\simeq[f(t)]^{-w}\Phi\{k/S[f(t)]\}, \quad S(t)\propto t^z, \tag{17}$$

where $f(t)$ is an unusual function of time, such as e^t, $\ln t$, 2^t, and so on. Thus we find that the exponents are

$$w=4IA_0, \quad z=2IA_0, \tag{18}$$

which imply that the exponents depend on the reaction rate I as well as the initial concentration A_0. Moreover, the total number decays as $\exp(-2IA_0t)$ while the characteristic aggregate size grows as $\exp(2IA_0t)$.

Now we turn to the general cases. Summing the governing rate equation (2), we obtain

$$\frac{dM_0}{dt}=-Ia_1M_\upsilon. \tag{19}$$

By analyzing all the above scaling solutions for $a_k(t)$ in the different cases, we find that $a_1(t)$ can be expressed in the form $a_1(t)=M_0^2(t)/M_1(t)$ and $a_k(t)$ ($k\gg1$) may be asymptotically written in a uniform form as follows:

$$a_k(t)\simeq\frac{[M_0(t)]^2}{M_1(t)}\exp(-x), \quad x=k\frac{M_0(t)}{M_1(t)}. \tag{20}$$

It is reasonable to assume that for general cases the solution of the rate equation (2) may also be written in the above scaling form of Eq. (20). Thus the problem reduces to finding the two moments $M_0(t)$ and $M_1(t)$. For our system, $M_1(t) \equiv A_0$. We cannot determine the exact solution of $M_0(t)$ and therefore turn to derive its asymptotic solution at large times. In the long-time limit, we can use the scaling form (20) to estimate $M_\upsilon(t)$ as

$$M_\upsilon(t) = \sum_{j=1}^{\infty} j^\upsilon a_j \simeq \left[\frac{M_1(t)}{M_0(t)}\right]^{\upsilon+1} \frac{[M_0(t)]^2}{M_1(t)} \int_0^\infty x^\upsilon e^{-x} dx$$
$$= \Gamma(1+\upsilon)[M_1(t)]^\upsilon [M_0(t)]^{1-\upsilon}, \tag{21}$$

which is valid in the case of $\upsilon > -1$. Inserting Eq. (21) into Eq. (19), we obtain

$$\frac{dM_0}{dt} \simeq -I\Gamma(1+\upsilon) M_1^{\upsilon-1} M_0^{3-\upsilon}. \tag{22}$$

When $\upsilon > 2$, one cannot obtain the solution of $M_0(t)$ from Eq. (22). It implies that the system may undergo a gelation-like transition in the $\upsilon > 2$ case. This case will invalidate the scaling form of Eq. (10) or (17). When $-1 < \upsilon < 2$, from Eq. (22) we derive the asymptotic solution of $M_0(t)$,

$$M_0(t) \simeq C_1 t^{-1/(2-\upsilon)}, \tag{23}$$

where $C_1 = [(2-\upsilon) I \Gamma(1+\upsilon) A_0^{\upsilon-1}]^{1/(\upsilon-2)}$. This shows that the total number of the aggregates decays as $t^{-1/(2-\upsilon)}$. For arbitrary exponent υ in the range of $-1 < \upsilon < 2$, we obtain the general scaling solution of $a_k(t)$ as follows:

$$a_k(t) \simeq A_0^{-1} C_1^2 t^{-2/(2-\upsilon)} \exp(-x), \quad x = A_0^{-1} C_1 k t^{-1/(2-\upsilon)}, \tag{24}$$

with the exponents

$$w = \frac{2}{2-\upsilon}, \quad z = \frac{1}{2-\upsilon}. \tag{25}$$

These show that the evolution behavior of our system obeys a quite different scaling law from that abided by preferential migration-driven aggregation growth in Ref. [14]. However, the typical aggregate size $S(t)$ grows as $t^{1/(2-\upsilon)}$ in the general case of $-1 < \upsilon < 2$, which is in agreement with the statement of the mean aggregate size for symmetric migration rate in Ref. [14].

We then investigate the case of $-2 \leqslant \upsilon \leqslant -1$. When $\upsilon = -1$, using ansatz (3) one can recast Eq. (2) to the following equations:

$$\frac{da}{dt} = -\frac{IA}{a}(1-a)^2 \ln(1-a),$$
$$\frac{dA}{dt} = \frac{2IA^2}{a}(1-a)\ln(1-a). \tag{26}$$

From Eq. (26) we determine the asymptotic solutions of $a(t)$ and $A(t)$ in the long-time limit,

$$a(t) \simeq 1 - (IA_0 t \ln t)^{-1/3}, \quad A(t) \simeq A_0 (IA_0 t \ln t)^{-2/3}. \tag{27}$$

Then we obtain an unusual scaling description for $a_k(t)$ at large times:

$$a_k(t) \simeq I^{-2/3} A_0^{1/3} (t \ln t)^{-2/3} \exp(-x),$$
$$x = k (IA_0 t \ln t)^{-1/3}. \tag{28}$$

This implies that the evolution behavior of the aggregate-size distribution obeys a logarithm-correction scaling law, and the exponents are $w = 2/3$ and $z = 1/3$. Moreover, the total number decays as $(t \ln t)^{-1/3}$ in this case. Similarly, in the $\upsilon = -2$ case, one can also obtain the asymptotic scaling solution

$$a_k(t) \simeq A_0 (C_2 t)^{-2/3} \exp(-x), \quad x = k (C_2 t)^{-1/3}, \tag{29}$$

where $C_2 = -3IA_0 \int_0^1 dx [\ln(1-x)/x]$. This shows that the aggregate-size distribution satisfies the usual scaling form of Eq. (10) with constant exponents $w = 2/3$ and $z = 1/3$. Meanwhile, the total number decays as $t^{-1/3}$, and its decay rate is less than that in the $\upsilon = -1$ case. As for the $-2 < \upsilon < -1$ case, we cannot find the explicit solution of the aggregate-size distribution; however, we can predict that the evolution of the system should be consistent with the generalized scaling form of Eq. (17), where the scaling time function $f(t)$ changes from $t \ln t$ to t. Moreover, all the systems with index υ in the range of $-2 < \upsilon < -1$ have the same universal exponents, $w = 2/3$ and $z = 1/3$.

When $\upsilon < -2$, we find $M_\upsilon \simeq C_3 [M_0(t)]^2$ (C_3 is a finite constant) in the long-time limit, and Eq. (19) then reduces to the asymptotic equation as follows:

$$\frac{dM_0}{dt} \simeq -C_4 M_0^4, \tag{30}$$

where $C_4 = IC_3 / A_0$. Equation (30) directly yields

$$M_0 \simeq (3C_4 t)^{-1/3}. \tag{31}$$

Thus, we find that for the $\upsilon < -2$ case the aggregate-size distribution also approaches the similar scaling form of Eq. (29). In order to confirm it, we investigate the last case, $\upsilon = -\infty$. In this case, the migration occurs only between the monomer aggregates and any other aggregates. Equation (2) is then rewritten as

$$\frac{da_k}{dt} = Ia_1 [(k+1)a_{k+1} + (k-1)a_{k-1} - 2ka_k]. \tag{32}$$

Under the monodisperse initial conditions, Eq. (32) can be solved exactly with the help of ansatz (3), and one can then find

$$a_k(t) \simeq A_0 (3IA_0 t)^{-2/3} \exp(-x), \quad x = k (3IA_0 t)^{-1/3}. \tag{33}$$

Indeed, Eq. (33) is also similar to the scaling solution (29) of the $\upsilon = -2$ case.

In summary, we have introduced a general migration-driven aggregation model with the symmetric migration rate kernel $K(k;j) = K'(k;j) = Ikj^\upsilon$. Based on the mean-field theory, we have analyzed the evolution behavior of the aggregate-size distribution in general cases with varying index υ. The results show that when $\upsilon \leqslant 2$, the evolution behavior of the system always obeys a scaling law different from that for the aggregate growth by preferential migration in Ref. [14]. Moreover, the aggregate-size distribution satisfies the conventional scaling form (10) when $-1 < \upsilon < 2$ or $\upsilon \leqslant -2$; and when $\upsilon = 2$ or $-2 < \upsilon \leqslant -1$, the evolution of the aggregate-size distribution obeys the generalized scaling form (17). The typical aggregate size grows as $\exp(2IA_0 t)$ in the $\upsilon = 2$ case while it grows as $t^{1/(2-\upsilon)}$ in the $-1 < \upsilon < 2$ case. The most interesting result is that for the $\upsilon \leqslant -2$ case the typical aggregate size always grows as $t^{1/3}$ and the aggregate-size distribution satisfies the similar scaling form with the same exponents. This model may be used to investigate the distribution of city populations as well as the evolution of the wealth distribution in economic activities.

This project was supported by the National Natural Science Foundation of China under Grant No. 10175008.

[1] R.L. Drake, in *Topics of Current Aerosol Research*, edited by G.M. Hidy and J.R. Brook (Pergamon, New York, 1972).

[2] S.K. Friedlander, *Smoke, Dust and Haze: Fundamental of Aerosol Behavior* (Wiley, New York, 1977).

[3] A. Pimpinelli and J. Villain, *Physics of Crystal Growth* (Cambridge University Press, Cambridge, U.K., 1998).

[4] P. Meakin, Phys. Rev. Lett. **51**, 1119 (1983); Rep. Prog. Phys. **55**, 157 (1992).

[5] T. Vicsek and F. Family, Phys. Rev. Lett. **52**, 1669 (1984); T. Vicsek, P. Meakin, and F. Family, Phys. Rev. A **32**, 1122 (1985).

[6] M.H. Ernst, in *Fundamental Problems in Statistical Physics VI*, edited by E.G.D. Cohen (Elsevier, New York, 1985).

[7] S. Song and D. Poland, Phys. Rev. A **46**, 5063 (1992).

[8] P.L. Krapivsky and S. Redner, Phys. Rev. E **54**, 3553 (1996).

[9] *The Economy As An Evolving Complex System*, edited by P.W. Anderson *et al.* (Addison-Wesley, Redwood, 1988).

[10] *The Theory of Income and Wealth Distribution*, edited by Y.S. Brenner *et al.* (St. Martin's Press, New York, 1988).

[11] F. Schweitzer, Adv. Complex Syst. **1**, 11 (1998).

[12] L.A.N. Amaral, P. Gopikrishnan, V. Plerou, and H.E. Stanley, Physica A **299**, 127 (2001).

[13] S. Ispolatov, P.L. Krapivsky, and S. Redner, Eur. Phys. J. B **2**, 267 (1998).

[14] F. Leyvraz and S. Redner, Phys. Rev. Lett. **88**, 068301 (2002).

[15] A.J. Bray, Adv. Phys. **43**, 357 (1994); D.H. Zanette and S.C. Manrubia, Phys. Rev. Lett. **79**, 523 (1997); M. Marsili and Y.C. Zhang, *ibid.* **80**, 2741 (1998).

[16] P.L. Krapivsky, Physica A **198**, 135 (1993).

[17] Jianhong Ke and Zhenquan Lin, Phys. Rev. E **65**, 051107 (2002).